Lecture 20 (11/10/21)

The Index of a closed curve.

p.w. smooth

Prop 1. Let γ be closed curve in $\mathbb{C}$. Then, $\frac{1}{2\pi i}\int_\gamma \frac{d\zeta}{\zeta - z}$ is an integer for all $z \notin \gamma$.

Pf. Assume $\gamma : [0,1] \to \mathbb{C}$ is smooth. Consider

$$g(t) = \int_0^t \frac{\gamma'(s)\,ds}{\gamma(s) - z}.$$ By FTC,

$$g'(t) = \frac{\gamma'(t)}{\gamma(t) - z}.$$ Consider $h(t) = e^{-g(t)}(\gamma(t) - z)$.

$$h'(t) = -g' e^{-g}(\gamma - z) + e^{-g}\gamma' = -\gamma' e^{-g} + \gamma' e^{-g} = 0.$$

Hence, h is constant and $h(1) = h(0) \Rightarrow$

$$\gamma(0) - z = e^{-g(1)}(\gamma(1) - z) \Rightarrow e^{-g(1)} = 1$$

$\Rightarrow g(1) = 2\pi i k,\ k \in \mathbb{Z}$, as desired.

The case γ is p.w. smooth is DIY. $\blacksquare$

Now, recall from Leibnitz rule that

$$n(\gamma, z) = \frac{1}{2\pi i} \int_\gamma \frac{d\zeta}{\zeta - z}$$

is continuous for $z \in G = \mathbb{C} \setminus \{\gamma\}$ (in fact, analytic by Leibnitz in Ex IV.2.2). Since, by Prop 1, it only takes integer values $\Rightarrow$

<u>Thm 1</u>. Let γ be closed curve. Then, $z \to n(\gamma, z)$ is an integer valued function on $G = \mathbb{C} \setminus \{\gamma\}$, which is constant on each component and $= 0$ on the unbdd component.

<u>Rem</u>. Recall that if γ is a curve then $-\gamma$ is the curve traversed in opposite direction. If σ starts where γ ends, then $\gamma + \sigma$ is the curve going from $\gamma(0)$ to $\gamma(1) = \sigma(0)$ to $\sigma(1)$. If γ and σ are closed, it is easy to see that

- $n(-\gamma, z) = -n(\gamma, z)$
- $n(\gamma + \sigma, z) = n(\gamma, z) + n(\sigma, z)$

In particular, traversing γ k times gives $n(k\gamma, z) = k\,n(\gamma, z)$.

Def. ① $n(\gamma, z)$ is the index or winding number of the closed curve γ w.r.t. $z \in \mathbb{C} \setminus \{\gamma\}$.

Cauchy's Integral formula – I. Let f be analytic in G, γ a closed curve in G s.t. $n(\gamma, z) = 0$ $\forall z \in \mathbb{C} \setminus G$. Then,

$$n(\gamma, z)\, f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)\, d\zeta}{\zeta - z}, \quad z \in G \setminus \{\gamma\}.$$

Moreover,

$$n(\gamma, z)\, f^{(k)}(z) = \frac{k!}{2\pi i} \int_\gamma \frac{f(\zeta)\, d\zeta}{(\zeta - z)^{k+1}}$$

CIF – II. If $\sum_{j=1}^{m} n(\gamma_j, z) = 0$ in $\mathbb{C} \setminus G$, then

$$\sum_{j=1}^{m} n(\gamma_j, z)\, f(z) = \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{\gamma_j} \frac{f(\zeta)\, d\zeta}{\zeta - z}.$$

Before proving CIF, we illustrate the two versions w/ an example

<u>Ex 1.</u> Let $f(z) = 1/z$ and $G = \mathbb{C} \setminus \{0\}$.

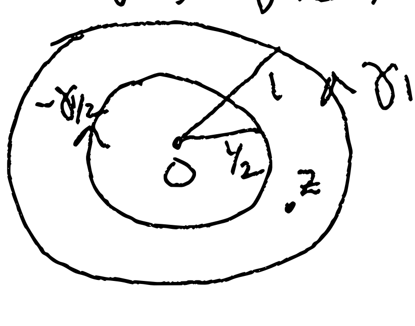
Then, for if γ_r denotes circle of radius $r > 0$ in pos. direction, then for $z \notin \gamma_r$:

$$\frac{1}{2\pi i} \int_{\gamma_r} \frac{f(\zeta)}{\zeta - z}\, d\zeta = -\frac{1}{z}\frac{1}{2\pi i}\int_{\gamma} \frac{d\zeta}{\zeta} + \frac{1}{z}\cdot\frac{1}{2\pi i}\int_{\gamma}\frac{d\zeta}{\zeta - z}$$

$$= -\frac{1}{z} n(\gamma, 0) + \frac{1}{z} n(\gamma, z) = \begin{cases} 0, & |z| < r \\ -1/z, & |z| > r. \end{cases}$$

Thus, $n(\gamma, z) f(z)$ is not given by this formula for any z and the assumption on γ, G is violated in CIF-I, since $n(\gamma, 0) = 1$.

Let us instead consider the pair of closed curves $\gamma_1, -\gamma_{1/2}$, and $\frac{1}{2} < |z| < 1$:



Now, $n(\gamma_1, 0) = 1$, $n(-\gamma_{1/2}, 0) = -1$
$\Rightarrow$ $n(\gamma, z) + n(-\gamma_{1/2}, z) = 0$ on $\mathbb{C} \setminus G = \{0\}$. Thus, CIF-II is valid and

$$f(z) = \frac{1}{z} = \frac{1}{2\pi i} \int_{\gamma_1} \frac{1/\zeta}{\zeta - z}\, d\zeta - \frac{1}{2\pi i} \int_{\gamma_{1/2}} \frac{1/\zeta}{\zeta - z}\, d\zeta,$$

for $\frac{1}{2} < |z| < 1$.

<u>Pf of CIF-I</u>. Consider the function in $G \times G$:

$$\varphi(z, \zeta) = \begin{cases} \dfrac{f(\zeta) - f(z)}{\zeta - z}, & \zeta \neq z \\ f'(z), & \zeta = z. \end{cases}$$

Since f is analytic ($\mathbb{C}$-diff.), φ is continuous and for each fixed ζ, $z \to \varphi(z, \zeta)$ is analytic. Why? (HW)
It is clearly analytic in $G \setminus \{\zeta\}$. Near the point ζ (in $B(\zeta, r) \subseteq G$), we may Taylor expand

$$f(z) = f(\zeta) + f'(\zeta)(z-\zeta) + (z-\zeta)^2 g(z),$$

where $g(z) = \sum_{n=0}^{\infty} \frac{f^{(n+2)}(\zeta)}{(n+2)!}(z-\zeta)^n$ is analytic

(ζ here is fixed in G). $\Rightarrow$

$$\varphi(z,\zeta) = f'(\zeta) + (z-\zeta)^2 g(z).$$

This is analytic. It the follows from $\mathbb{C}$-Leibniz that

$$\psi(z) = \frac{1}{2\pi i}\int_\gamma \varphi(z,\zeta)\,d\zeta = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)\,d\zeta}{\zeta - z}$$

(*)

$$+ f(z)\frac{1}{2\pi i}\int_\gamma \frac{d\zeta}{\zeta - z} = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)\,d\zeta}{\zeta - z} + n(\gamma, z) f(z)$$

is analytic in G. Moreover, the function

$$\tilde{\psi}(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)\,d\zeta}{\zeta - z}$$ is analytic in

$\mathbb{C} \setminus \gamma$ (also by $\mathbb{C}$-Leibniz).

Since $n(\gamma, z) = 0$ on $\mathbb{C} \setminus G$, we can extend φ to all of $\mathbb{C}$ by defining it as $\tilde{\varphi}(z)$ on $\mathbb{C} \setminus G$. To see that this is analytic, i.e. <u>entire</u>, we let $H = \{z \in \mathbb{C} \setminus \gamma : n(\gamma, z) = 0\}$. By Thm 1, H is open, consisting of components of $\mathbb{C} \setminus \gamma$, and by assumption $\mathbb{C} = G \cup H$. Since both $\varphi, \tilde{\varphi}$ are analytic where defined it suffices to check that φ and $\tilde{\varphi}$ coincide on $G \cap H$. But this is obvious by $(*)$.

We claim that $|\varphi(z)| \to 0$ as $z \to \infty$, which finishes the pf by Liouville's Thm. Since H contains the unbdd component of $\mathbb{C} \setminus \gamma$, we may compute $\lim_{z \to \infty} |\varphi(z)|$ by using $\tilde{\varphi}(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta$. This is easily seen to go to 0 as $z \to \infty$ since

$$|\tilde{\varphi}(z)| \leq \frac{1}{2\pi} \sup_{\zeta \in \gamma} |f(\zeta)| \cdot L(\gamma) \cdot \frac{1}{|z| - \sup_{\zeta \in \gamma} |\zeta|} \to 0.$$

$\boxtimes$